THE MATHEMATICS OF DECISION-MAKING

by

DR. FEDERICO M. SIOSON*

That mathematics is a tool for decision-making is well-known. Not too long ago a group of American and German educators proposed a degree in the essential preparation for executives. More and more scientists are becoming aware of the fact that no form of human knowledge can have any claim to permanence or timelessness unless it can render itself amenable to the cold and sharp analysis of symbols and axiomatics. Plato himself expressed this undying faith in the power of mathematical reasoning when he hung on the portals of his Academy the admonishing words: "Let no ignorant geometry enter here". To the French philosopher, Rene Descartes, everything inevitably turns into mathematics. The great English physicist, Lord Kelvin, once said that if you could speak of a phenomenon but could not represent that phenomenon in the language of symbols, then you could not say you know anything about it; if you could, then you might say you know something about it.

Strictly speaking, the problem of decision making is essentially a problem of logic. In fact, the first formal decision problems were formulated and solved in the domain of mathemtecal logic itself. While perhaps some of these solved decision problems are the finest and profoundest examples of human cerebration, their exposition is out of the question in a paper such as ours. Specifically, the problem we would like to consider may be formulated as follows:

Statement of the Problem: Given any class A of possible courses of action and a certain utility index or function f defined over these various courses of action and whose values are

* Formerly Assistant Professor, Ateneo de Manila University.

(partiallly) ordered, the fundamental problem is to determine that course of action or those courses of action m which maximize or minimize the utility index. In formal symbols, this is the determination of an element m of A such that $f(m) \geqq f(x)$ or $f(m) \leqq f(x)$ for all possible members x of A.

Quite often, the most difficult aspect of this problem is to find the appropriate utility index for a specific case. The literature on the subject abounds in many differing views concerning the notion of utility and its axiomatic foundation. The finer and more subtle points of the concept have led to some questions of a highly philosophical nature that we cannot enter into here. Suffice it to say, utility, whether in terms of money, expected value, degree of moral obligation or commitment, will be assumed as something familiar to everyone of us. Sometimes, we shall treat it like any arithmetical quantity which may be added, subtracted, multiplied,and divided; at other times, we shall not and the only property of it that we will accept is it partial ordering. A warning therefore at this point must be given the listener who may refer to the existing literature: our notion of utility is a little loose and may not coincide with the accepted one.

There are no known general methods that can effect a universal solution of the main problem of decision-making stated above. Two well-known theories, the theory of games and the theory of linear programing, however, turns out to be very usefull in solving a very large variety of decision problems of the type mentioned. These are correspondingly discussed in some detail in the following sections.

1. The Theory of Linear Programing.

Linear programming is primarily designed to solve the following general mathematical problem: To determine a set of n real numbers $x_1, \ldots, x_n$ optimizing (i.e. maximizing or minimizing) a linear function $f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ subject to the restricting conditions:

$$x_1 \geqq 0, \quad x_2 \geqq 0, \ldots, x_n \geqq 0,$$

$$b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n \geqq c_1,$$

$$b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n \geqq c_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$b_{m1}x_1 + b_{m2}x_2 + \ldots + b_{mn}x_n \geqq c_m.$$

The fact that there are no restricting conditions of the form

$$b_1x_1 + b_2x_2 + \ldots + b_{1n}x_n \leqq c$$

in this formulation is no less of generality, since any inequality of this form may be reduced into the desired form by merely multiplying both sides of the inequality by -1, i.e.

$$(-b_1)x_1 + (-b_2)x_2 + \ldots + (-b_n)x_n \geqq -c.$$

Also, the case of equality is adequately covered by the present formulation, for any quality $b_1x_1 + b_2x_2 + \ldots + b_nx_n = c$ is in fact equivalent to the inequalities $b_1x_1 + b_2x_2 + \ldots + b_nx_n \leqq c$ and $b_1x_1 + b_2x_2 + \ldots + b_nx_n \geqq c$.

It is easy to see that linear programming is indeed a special case of the fundamental problem of decision-making stated above, where the set of all possible courses of action A consists of all n-tuples of real numbers $(x_1, \ldots, x_n)$ subject to the restricting inequalities and

$$\text{where} \quad f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n$$

denotes the utility index.

Mathematically, the problem of linear programming is a problem of maximization or minimization, but the methods of infinitesimal and variation calculus cannot be used in handling it. The maximum and minimum points in programming problems often lie on the boundary of its domain of definition and not in its interior. The theory of convex sets and topology proved to be more important tools in searching for their solutions.

Let us present a brief and geometrical account of this method in 3-dimensional space. A set of points in 3-space is said to be convex if and only if the segment joining any two points of the set wholly belong to the set. Examples of convex sets are the collections of all points inside a sphere, a cube, a square, a polygon, a cylinder, and line segment. It may be easily shown that the set $C_1 \cap \ldots \cap C_k$ of points common to a number of convex sets $C_1, \ldots, C_n$ (usually called their intersection) is always a convex set. For, suppose P and Q are any two points of the intersection; then by definition of convexity both P and Q must belong to each of the convex sets $C_1, \ldots, C_n$. Hence, the line segment joining P and Q must be totally contained in each of the convex sets $C_1, \ldots, C_n$ and therefree in their intersection too.

After these preliminaries let us now consider the linear programming problem in 3-dimensions: To determine a point (x,y,z) in 3-space that optimizes, say minimizes, the linear function $Ax + By + Cz$ subject to the linear inequalities

$$x \geqq 0, \quad y \geqq 0, \quad z \geqq 0,$$
$$a_1x + b_1y + c_1z \geqq d_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_mx + b_my + c_mz \geqq d_m.$$

The collection of all points (x.y,z) satisfying any one of these inequalities are points lying on and on side of a plane and hence a convex set. Consequently, the collection of all points that satisfy all the above inequalities which is the intersection of all those convex sets must also be a convex set, in fact, a convex polyhedron. Geometrically speaking, the problem is then to determine the coordinates of the vertex, edge, or face of the polyhedron lying on a plane $Ax + By + Cz = D$ nearest to the origin. The general linear programming problem is analogously the determination of that hyperface of a convex hyper-polyhedron in many-dimensional space lying on some hyperplane nearest to the origin.

There are a number of well-known methods of effecting such a determination, but these fall outside the realm of our present interests. Specific examples will be given and solved in a manner we deem appropriate.

2. The Theory of Games.

The theory of games is a formal mathematization of conflict and sometimes by chance too. This is the case, for instance is partly controlled by each of the opposing parties in the conflict and sometimes by chance too. This is the case, for instance, in parlor games such as chess, poker, bridge, and canasta, and in many situations in war, politics, ethics, and economics. Game theory goes beyond the classical theory of probability by emphasizing the strategic aspects of conflict raher than its aspect due to pure chance.

Consider a very simple example: the game where player P_1 picks a number from a collection consisting of 1, 2, and 3 and where player P_2 picks one of the number 1, 2, 3, and 4. after their choices have been made simoltaneouly P_2 then pays P_1 the amount specified by the following (payoff) matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

This matrix signifies, for instance, that if P_1 choose i, and P_2 choose j then P_2 pays P_1 the amount given by a_{ij}. When a_{ij} is negative, what actually happens is that P_1 pays P_2 the amount equal to the absolute value of a_{ij}.

The immediate query whose answer we want is: What is the best possible way for P_1 or P_2 to play this simple game? Surely, if P_1 knows thath P_2 is choosing 4 and a_{m4} is the largest of the numbers a_{14}, a_{24}, a_{34}, then P should choose m. But the numbers are to be chosen by P_1 and P_2 independently and simultaneously so that neither one of them knows what the other is choosing.

One may look at it, however, in this way, If P_1 happens to choose 1, the worst that can happen to him is to be paid only $\min_j a_{1j}$, the minimum or least of the numbers a_{11}, a_{12}, a_{13}, and a_{14}. Analogous remarks apply when P_1 happens to choose 2 or 3. A possible strategy P_1 may then use is to pick that number which realizes the best of his worts payoff, that is, $\max_i \min_j a_{ij}$. This is the so called maximin strategy of P_1. Under this strategy, player P_1 is thus assured of getting at least the amount $\max_i \min_j a_{ij}$. Remembering that the payments of P_1 to P_2 are just the negatives of the payments of P_2 to P_1, it follows then that P_2 can also ensure himself of getting at least the amount $\max_j \min_i -a_{ij} = -\min_j \max_i a_{ij}$ and by so doing he can also prevent P_1 from getting more than $\min_j \max_i a_{ij}$. Hence if it happens that $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$, then player P_1 can realize the value $v = \max_i \min_j a_{ij}$ and the best that P_2 can do is to prevent him from getting more than v. from the point of view of P_2 ,he can realize the value -v and the best that P_1 can do is to prevent him from getting more than -v. Unless there is therefore any reason to believe that either one of them is going to play wild or that one of them has fixed habits, then thebest strategy for both P_1 and P_2 is to play those numbers i*, j* respectively such that $a_{i^*j^*} = \max_i \min_j a_{ij} = \min_j \max_i a_{ij}$. They are called the optimal pure stratgies of P_1 and P_2 respectively. The pair (i*, j*) is also called a saddle point since $a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j}$ for each $i = 1, 2, 3,$ and each $j = 1, 2, 3, 4$.

It occurse on the matrix where the entry is the minimum of its row and the maximum of its column. If the game mentioned above, for example, has the specific payoff matrix

$$\begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -6 & -2 & 0 & -5 \end{pmatrix}$$

then the optimal pure strategy for P_1 is to choose 2 and 3 for P_2, inasmuch as (2 , 3) is a saddle point. Of course, it is quite clear in this instance that P_2 is going to lose, but under the circumtances he can really not do anything better without the danger of incurring a greater loss. We may say that the game is simply rigged against him.

It remains still to propose an optimal strategy for playing a rectagular game with no saddle points like

$$\begin{pmatrix} 2 & 6 \\ 7 & 3 \end{pmatrix} .$$

Should player P_1 decide to use his maximin strategy, then he would he assured of getting at least $\max \left\{ \min_j a_{ij}, \min_j a_{2j} \right\} = \max \left\{ 2, 3 \right\} = 3$. P_1 may realize this by playing 2. On the other hand, suppose P_2 were to play also 2 which is his maximin strategy; then clearly P_1 could do better playing 1, for then he would realize a payoff of 6. But, of course, if P_2 knows that P_1 is playing 1, then he could do better by playing also 1; for in that case, P_2 would have to only pay P_1 2. and naturally, if P_1 is aware that P_2 is going to play 1, he should play 2 to realize the payoff 7. And so on. This argument has thus led us to a vicious circle.

Inasmuch as each player would like to play in such a manner that the other cannot anticipate or guess his moves, it would rather seem that the best thing for player P_1 to do is to randomize his choices and so with P_2. Let x and 1-x be the probalities of P_1 choosing respectively 1 and 2 and y 1-y be the probabilities of P_2 choosing 1 and 2 respectively. In this case, the mathematical expectation of P_1 will be

$$E(x,y) = 2xy + 6x(1-y) + 7(1-x)y + 3(1-x)(1-y)$$
$$= -8xy + 3x + 4y + 3 = -8(x - \tfrac{1}{2})(y - \tfrac{3}{8}) + \tfrac{9}{2}.$$

Hence, if P_1 picks that randomized strategy such that $x = 1/2$ he can be sure that his expected payoff will be at least 9/2. By picking that randomized strategy such that $y = 3/8$, P_2 can also prevent P_1 from increasing his expected payoff to more than 9/2. P_1 might just as well settle himself to playing 1 and 2 with the same probability 1/2 and P_1 might just as well play 1 and 2 with respective probabilities 3/8 and 5/8. The pair of probabilities (1/2, 3/8) is actually a saddle point of the expectation function E(x,y). The quantity $v = E(1/2, 3/8)$ is called the value of the game and the probability pairs (1/2, 1/2) and (3/8, 5/8) the optimal mixed strategies of P_1 and P_2 respectively.

The ideas treated in the previous paragraph may be generalized to any m by n matrix game. For any rectangular matrix game, the fundamental theorem of matrix game theory states that there always exist mixed optimal strategies for both players. If (a_{ij}) is any such m by n matrix game, then the value v of the game and the optimal mixed strategies $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ of players P_1 and P_2 respectively are in fact solutions of the following set of inequalities:

$$x_1 + x_2 + \ldots + x_m = 1, \qquad y_1 + y_2 + \ldots + y_n = 1$$

$$x_1 \geqq 0,\ x_2 \geqq 0,\ \ldots,\ x_m \geqq 0 \qquad y_1 \leqq 0,\ y_2 \leqq 0,\ \ldots,\ y_n \leqq 0,$$

$$a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \geqq v, \qquad a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leqq v,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_n \geqq v, \qquad a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leqq v.$$

As a first illustration, consider the well-known game of Paper, Stone, and Scissors. In this game both players must simultaneously name one of the objects stone, paper, or scissors. Paper defeats stone (since stone may be wrapped with paper), stone defeats scissors (since stone may be used for hammering scissors), and scissors defeat paper (since scissors may cut paper). The player who chooses the winning object say wins a peso; if both players choose the same object, the game is a draw.

The payoff matrix for this game is then

	P	St	Sc
P	0	1	-1
St	-1	0	1
Sc	1	-1	0

The corresponding mixed strategies (x_1, x_2, x_3) and (y_1, y_2, y_3) for players P_1 and P_2 are the solutions of the following set of inequalities and equalities:

$$x_1 + x_2 + x_3 = 1, \qquad y_1 + y_2 + y_3 = 1,$$
$$- x_2 + x_3 \geqq v, \qquad y_2 - y_3 \leqq v,$$
$$x_1 - x_3 \geqq v, \qquad - y_1 + y_3 \leqq v,$$
$$- x_1 + x_2 \geqq v, \qquad y_1 - y_2 \leqq v.$$

By considering the case when the inequalities are in fact equalities, it is easily seen that

$$x_1 = x_2 = x_3 = 1/3, \quad v = 0, \quad y_1 = y_2 = y_3 = 1/3$$

constitute a solution. The optimal way of how to play this game therefore is to choose paper, stone, or scissors randomly each with probability 1/3.

As another example of a game of strategy, let us consider the ancient Italian game of Two Finger Morra. This game is played by two persons, each of whom shows one or two fingers simultaneously calls his guess of the number of fingers shown by his opponent. If a player guesses correctly, he wins an amount equal to the sum of fingers shown by himself and his opponent; otherwise the game is a draw. If (i,j) denotes the instance when a player shows i fingers and guesses that his opponent shows j fingers, then the payoff matrix is given by

	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	0	2	-3	0
(1,2)	-2	0	0	3
(2,1)	3	0	0	-4
(2,2)	0	-3	4	0

By trial and error, one finds the solvable combination of equalities and strict inequalities which determine the value and optimal mixed strategies are

$$x_1 + x_2 + x_3 + x_4 = 1, \qquad y_1 + y_2 + y_3 + y_4 = 1$$

$$-2x_2 + 3x_3 > v, \qquad 2y_2 - 3y_3 < v,$$

$$2x_1 - 3x_4 \geqq v, \qquad -2y_1 + 3y_4 \leqq v,$$

$$-3x_1 + 4x_4 \geqq v, \qquad 3y_1 - 4y_4 \leqq v,$$

$$3x_2 - 4x_3 > v, \qquad 3y_2 - 4y_3 < v.$$

It is not difficult to verify that any quadruple (0, p, 1-p, 0) constitutes an optimal mixed strategy for either player as long as

$$20/35 \leqq p \leqq 21/35.$$

A distinct and important type of a two-person game is one where player P_2 is Mother Nature. Nature, of course, cannot be considered as a conscious opponent like a human being who takes advantage of our mistakes. For this reason, playing with Nature requires a different method of approach. There are four known criteria for a good strategy against Nature, but none of these can be said to be better than the others. Let us describe each of these criteria in the context of the following matrix game against Nature with states (Nature's strategies) s_1, s_2, s_3, and s_4:

	s_1	s_2	s_3	s_4
1	0	0	-2	-1
2	-1	-1	-1	-1
3	-2	2	-2	-2
4	-1	1	-2	-2

C_1. **The Maximum Criterion.** The good strategy for this criterion is to choose that course of action which will maximize the smallest payoffs corresponding to each of the possible courses of action. The respective minima, for example, of each of the four choices 1, 2, 3, and 4 are respectively -2, -1, and -2. The maximin strategy advises that choice which assures one of a payoff equal to the maximum of the minima -2, -1, -2, and -2. This choice is therefore no other than 2.

The maximin creterion is perhaps the safest and most pessimistic one of them all. It is the best answer to Mother Nature's most unfavorable probability distribution function over her states or strategies.

C_2. **The Minimax Regret Criterion.** If, in the game mentioned above, s_2 is the actual state of Nature, then P_1 may be said to have no regret in choosing 3. He will, on the other hand, have regrets if he chooses either 1, 2, or 4. A measure of this regret is then the maximum payoff in the s_2-column minus the correspoding entry in that column. Extending this idea to each of the other columns or states of Nature, we will obtain the following (regret) matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The good strategy proposed by the minimax regret criterion is that one which minimizes the largest payoffs in each row of the regret matrix. This criteria thus advises P to choose strategy 4 over all others.

In general, the regret payoff matrix of an m by n rectangular matrix game (a_{ij}) is the matrix whose (i,j)th entry is the number $\max_k a_{kj} - a_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

C_3. **The Laplacian Criterion.** The criterion of Laplace asserts that inasmuch as one does not know in any way which state of Nature is true, the most logical alternative is to assume that each state is equally likely to occur as any other. The best strategy under these circumstances of player P_1 in an m by n matrix game (a_{ij}) is to choose that row i whose expectation

$$\frac{a_{i1} + a_{i2} + \cdots + a_{in}}{n}$$

is maximum. For the game under discussion, this strategy is given by 1, since the sum of the elements in the first row of the given payoff matrix is obviously larger than the others.

C_4. **The Criterion of Hurwicz.**

Hurwics suggested a modification of the maximin criterion that partly eliminates its extreme pessimism. By fixing a certain index r between O and 1, the various values $r \max_j a_{ij} + (1 - r) \min_j a_{ij}$ of each row i of an m by n matrix game (a_{ij}) is considered. The row for which this value $r \max_j a_{ij} + (1 - r) \min_j a_{ij}$ is maximum is chosen as the Hurwicz best strategy.

In the example above, the four rows of the payoff matrix gives

$-2r + 0(1-r) = -2r$, $-r - (1-r) = -1$, $2r - 2(1-r) = 4r - 2$, and $r - 2(1-r) = 3r - 2$ respectively. Whence, if r is greater than $1/4$, then 3 is the good strategy according to criteria of Hurwicz with index r.

The games we have considered so far are matrix games. One may feel that these are very restricted forms of games, but this is far from being so. It can be shown that any game of strategy played by two persons can actually be normalized into a matrix game whose strategies consist of all possible ways of playing the game from beginning to end. For some of the common games like chess and bridge, the payoff matrices may run to millions and consequently, they are still beyond the power of even the most powerful computing machines at present to solve.

The rest of the communication will now be devoted to applications.

3. **Applications to Military Science.**

There are two philosophies on which a military tactician may base his course of action: on what his enemy is capable of doing or on what his enemy is going to do. An officer in the United States Armed Forces is for instance enjoined to choose that decision which offers the greatest probability of success in view of what the enemy can do, not on what he things the enemy is going to do. When viewed thus as a two-person game the American philosophy of military action is actually the maximin-maximax principle. Let us examine this principle in connection with an actual case that happened during the last World War in the well known Battle of the Bismark Sea.

Once during the New Guinea campaign, American intelligence reported that the Japanese would move a supply and troop convoy from the port of Rabaul, New Britain to a place called Lae just west of New Britain, on the eastern tip of New Guinea. The Japanese convoy could either travel north of New

Britain where it is certain there would be poor visibility or south of the island where the weather would be clear. In either case the trip will take three days. The American commander, General Kenney, then had two alternatives: either to concentrate the bulk of his reconnaisance aircraft on the northern route or on the southern route, Once sighted, the convey could be bombed until its arrival at Lae. General Kenney's staff estimated the following possible outcomes measured in bombing days:

		Japanese	
		Northern	Southern
American	Northern	2	2
	Southern	1	3

Thus, by making use of the ideas of game theory General Kenney decided to concentrate his reconnaisance planes in the northern route, since (1,1) and (1,2) are the saddle points of the above payoff matrix. The convoy was in fact sighted after one day and the Japanese suffered heavy losses. Inspite of this, the Japanese commander, it must be emphasized, did not err in his decision. Under the circumstances, he had chosen the best strategy. In fact, the other strategy open for him could have inflicted direr consequences.

The following is another example of how military decision making is done by mathematical methods. Suppose that the strategic air command is instructed to cripple the enemy's oil production. This may be done by destroying the enemy's only oil refinery or its only oil field. Now, there is an acute shortage of plane fuel which therefore limits the supply for this mission to only 24,000 gallons. Any bomber sent to the mission must have enough fuel for the round trip plus a reserve of 100 gallons. There are two types of bombers available and their descriptions are as follows:

Bomber Type	Description	Miles/Gallon	Number of Planes
1	Heavy	2	24
2	Medium	3	16

Army intelligence sources have made available the following information the enemy's oil field and refinery.

		Probabiltiy of Success	
	Distance	Heavy Bomber	Medium Bomber
Oil Field	510	.15	.10
Oil Refinery	600	.25	.20

The problem is how many of each of these bombers should be dispatched to the oil field and oil refinery in order to maximize the probability of destroying at least one of the plants: the oil refinery or the oil field. This problem is of course equivalent to the problem of minimizing the probability of not destroying any plant. Assume that no damage is inflicted on either plant by a bomber that fails to destroy it.

If x_{11} and x_{12} denote the number of heavy bombers that should be sent to the oil field and oil refinery respectively and x_{21} and x_{22} the same number of medium bombers that should be sent to the oil field and refinery respectively, then our problem is to minimize the probability

$$P = (1 - .15)^{x_{11}}(1 - .25)^{x_{12}}(1 - .10)^{x_{21}}(1 - .20)^{x_{22}}$$

or

$$-\log P = .07041x_{11} + .12483x_{12} + .0457x_{21} + .09691x_{22}$$

subject to the restrictions on available fuel and planes given by the following inequalities:

$$x_{11} + x_{12} \leqq 24,$$

$$x_{21} + x_{22} \leqq 16$$

$$\frac{2(510)}{2}x_{11} + \frac{2(600)}{2}x_{12} + \frac{2(510)}{3}x_{21} + \frac{2(600)}{3}x_{22} + 100(x_{11} + x_{12} + x_{21} + x_{22}) \leqq 24,000$$

or

$$610x_{11} + 700x_{12} + 440x_{21} + 500x_{22} \leq 24{,}000.$$

It is easily verified that a solution of the problem is given by $x_{11} = x_{21} = 0$, $x_{12} = 24$, $x_{22} = 16$. This means th the heavy and medium bombers should be dispatched to the oil refinery.

Heterogeneity of weapons in any defense air system is necessary. This enables one to cope with the gamut of possible attack strategies the enemy may use. The problem connected with decision-making in such situations may also be mathematized. Consider a case of two mobile air defense missile systems, with missile system M especially effective against high altitude attacks and missile system M' being most effective against low altitude attacks. The mathematical problem is to determine the relative proportions of M and M' that will provide the optimally effective defense.

Tactically, the situation is as follows: a point target (say an important arsenal) is threatened with attack by fighter- bombers. The attack may oncentrate eitehr on low altitude ($j = 1$) or on high altitude ($j = 2$). Due to some navigational and coordination problems, low altitude attacks are more costly than high altitude attacks. For this reason, assume that the enemy can at most only deploy 14 low altitude aircraft or at most 20 high altitude aircraft. On the other hand, low altitude aircraft pose more threat and its destructive potential $e_1 = 1$ is twice that of high altitude aircraft which is $e_2 = 1/2$. The missile systems are supposed to be deployed along circles around the target and for each radius R from the point target there is associated a certain kill potential $K_{ij}(R)$ which depends on the mode of attack j and type of defense i. The kill potential for each unit of weapon corresponding to the radii 0, 10, and 20 miles is given by the following table:

Deployment Radius (R_i)	Low altitude attack (j=1)	High altitude attack (j=2)	Low altitude attack (j=1)	High altitude attack (j=2)
0	0	6	0	0
10	1	2	3	1
20	1/2	1	1/2	0

The other radii like the last are of no significance in the defense of the target, since their consideration will add no dominating strategies in the game that will be considered.

Weapon M costs 50 thousand dollars each and weapon M' costs 25 thousand dollars each and the total operational budget is 100 thousand dollars. There are then only three feasible weapon combinations under the conditions:

Weapon Combination (k)	Weapon M Force Level	Weapon M' Force Level
1	2	0
2	1	2
3	0	4

Since also the only significant and tenable radii of deployment are either $R = 0$ or $R = 10$, then the only two possible deployment for the available weapons are:

$y = 1$: Weapon M at $R = 0$ and weapon M' at $R = 10$

and

$y = 2$: Weapon M at $R = 10$ and weapon M' at $R = 10$.

The payoff matrices associated with each weapon combination k and each weapon deployment y is computed as

$$(a^k_{yj}) = (e_j(m_j - n^k_M \cdot K_{Mj}(R^y_M) - n^k_{M'} \cdot K_{M'j}(R^y_{M'}))),$$

where $m_1 = 14$, $m_2 = 20$, $e_1 = 1$, $e_2 = 1/2$, n^k_M = the number of weapons M in the weapon combination k, $n^k_{M'}$ = the number of weapons M' in combination k, $K_{Mj}(R^y_M)$ = the kill potential relative to the mode of attack j and type of defense M deployed according to y at a radius R^y_M from the target, and $K_{M'j}(R^y_{M'})$ = the kill potential relative to j and defense M' deployed according to y at a radius R^y_M from the target. Hence, the possible payoff matrices corresponding to the various weapon combinations are:

		$j = 1$	$j = 2$
$k = 1$:	$y = 1$	14	4
	$y = 2$	12	8
$k = 2$:	$y = 1$	8	6
	$y = 2$	7	8
$k = 3$:	$y = 1$	2	8
	$y = 2$	2	8

The values, optimal defense and attack strategies of these matrix games are given by

k	Optimal Defense Strategy	Optimal Attack Strategy	Value
1	(0,1)	(1,0)	12
2	$(\frac{1}{3},\frac{2}{3})$	$(\frac{2}{3},\frac{1}{3})$	$\frac{22}{3}$
3	(p,1-p)	(0,1)	8

In words, this means that for the two-zero combination of weapons M and M' the best pure strategy for defense is to deploy weapons M on the rim of the circle of radius 10 miles from the target. To find the weapon combination that minimizes the damage to the target, one selects that weapon combination with the biggest value, that is, $k = 2$. Thus the recommended defense that will minimize target damage is to use two of M' and one of M.

4. **Application to Ethics.**

We shall consider the following ethical problem. Suppose that two friends of long standing, say Juan and Pedro, hold positions of the same rank in a corporation. Juan through some means knows that the vice-presidency is going to be vacated and filled by either him or Pedro. He also knows that the actual promotion will hinge on the final judgement of the corporation's president. Must Juan pass the information to Pedro?

The utility index in this problem may be taken as the total good (the summum bonum) that can be achieved. The material benefit p that will accrue to Juan if he gets the promotion is equal to that of Pedro if Pedro gets the promotion. Clearly, there are two possible courses of action for Juan: either to tell Pedro or not of the impending promotion. If Juan tells Pedro about it, then certainly Juan will retain the friendship of Pedro and the corresponding utility index value would be f(Juan informs Pedro) = p. However, if Juan decides to keep the secret to himself, then he loses the friendship of Pedro even if Pedro never learn of Juan's actuation. Assuming that this lost friendship is a quantity q, then the corresponding utility value for this course of action is f(Juan does not inform Pedro) = P — q. It is quite clear therefore that under this utilitarian criterion the best course of action for Juan is to inform Pedro.

Using the egoistic or hedonistic criterion, Juan may feel that if Pedro does not know of the promotion, then he is in a better position to get the promotion. In this instance, Juan's optimal course of action would depend on which of the utility values

$$f(\text{Juan informs Pedro}) = \text{Expectation of Juan} = p/2$$

$$f(\text{Juan does not inform Pedro}) = p\text{-}q$$

is bigger. To many people q is worth so much more than any financial consideration that even in this instance the first course of action is the optimal way.

The following is another example of an ethical game borrowed from R. B. Braithwaite's book, The Theory of Games as a Tool for the Moral Philosopher.

Suppose Luke and Matthew are two bachelor occupying a duplex house with very poor accoustics so that Luke hears everything louder than a conservation that takes place in Matthew's flat and vice versa. Suppose further that each of them has only the hour from 9:00 to 10:00 P.M. for recreation and no other. Luke's form of recreation is to play classical music on the piano for an hour without pause and Matthew's amusement is to improvise jazz on the trumpet for an hour at a time too. That whether or not one of them performs on any evening has nothing to do with the desires of the other to perform on any evening. The satisfaction derived from playing be each of them affected of course by whether the other is playing or not, but in this instance what one gets does not come from the other; in other words their game is no longer zero-sum. One may write a possible payoff matrix for their game as:

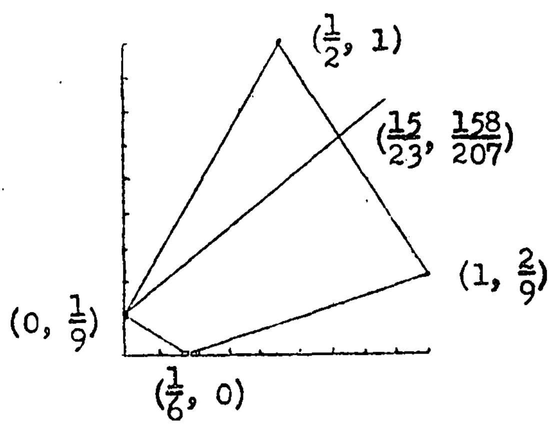
		Matthew	
		plays	does not play
Luke	plays	(1,2)	(7,3)
	does not play	(4,10)	(2,1)

For example, (4,10) here indicated that Luke gets 4 utiles of satisfaction and Matthew 10 utiles in the event that Luke does not play and Matthew plays. The problem is to determine the best palusible way of settling the ethical dispute between them.

The method of solution suggested by Howard Raiffa starts by transforming the (utility) payoff matrix into a canonical form such that 0 is the worst payoff of anybody and 1 his best. If this procedure were then applied to the above payoff matrix, the new payoff matrix becomes

$$\begin{pmatrix} (0, \frac{1}{9}) & (1, \frac{2}{9}) \\ (\frac{1}{2}, 1) & (\frac{1}{6}, 0) \end{pmatrix}$$

If these tranformed payoff values are plotted in a rectangular coordinate system, then one would obtain as possible payoff values for the game the set of all points inside the following polygon:



Each pair of mixed strategies (x,1-x), (y,1-y) for Luke and Matthew respectively will achieve a point in this polygon as payoff and conversely, each point-payoff inside the above shaded area can be achieved by some pair of strategies as long as the game is rendemized or played randomly many times.

If (a,b) and (a′, b′) are any two point-payoffs in the convex polygon above such that a is greater than or equal to a and b is greater than or equal to b′, then (a,b) is a more desirable payoff to achieve. Thus Luke and Matthew need not play to achieve any point-payoff jointly dominated by some other payoff. The undominated point-payoffs of a game is called the Pareto optimal set. In the so-called von Neuman-Morgenstern theory, these point-payoffs constitute the solutions of the game. The set of all points between and including (1/2,1) and (1,2/9) is the Pareto optimal set for Luke and Matthew's game.

Obviously, Luke desires (1/2, 1) most, i.e., that he plays and Matthew remains silent, while Matthew wants (1,2/9) most. None of them, however, can realized his heart's desire without the other's knowing and altruism. If Luke and Matthew were to both play, their payoff would only be (0, 1/9) which is clearly not the most desirable thing for both of them. Of course, should both of them play their maximin strategies, then Luke will guarantee himself a payoff of 1/6 and Matthew will be assured of a 1/9 payoff, but not one of them would get the most of what he can. How should they cooperate for their mutual benefit.

The suggestion is that they first play the zero sum (competitive) game whose payoff matrix consists of their relative payoff advantages and then resolve that they fully cooperate with one another to increase their payoffs as much as possible while preserving their relative advantages.

The relative advantages of Luke over Matthew for each of the four possible act-combinations is given by the matrix

$$\begin{pmatrix} -\frac{1}{9} & \frac{7}{9} \\ -\frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

It is easily seen that (1,1) is a saddle point and hence the corresponding value for this game of relative advantages is (0,1/9). Now, any point on the line passing through this point with slope 1 will represent a payoff with the same relative advantage of $-\frac{1}{9}$ for Luke. The good cooperative strategy for Luke and Matthew is to attempt to achieve the payoff

$$(15/23,\ 158/207 = \frac{16}{23}(1/2,\ 1) + \frac{7}{23}(1,\ 2/9),$$

the point on the Pareto optimal set intersected by the line of relative advantage $-\frac{1}{9}$ in favor of Luke. The suggested arbitration then should be that Matthew should play while Luke remains silent 16 out of 23 nights and Luke should play while Matthew be silent 7 out 23 nights.

There are other suggested solutions to this game, but a discussion of them here will sink us into theory deeper than we want to be.

5. **Applications to Statistics.**

Any statistical decision problem is a two person game with Nature as one player and the statistician as the other. The statistician in a statistical game spies on Nature by performing experiments. Thus in the background of any statistical game is a sample space X which describes all possible outcomes of his experiments. Associated with the sample space X is a collection of distribution functions p_θ each determined by one more parameters θ such that

$$\sum_{x \in X} p_\theta(x) = 1 \quad \text{or} \quad \int_X p_\theta(x)dx = 1.$$

Any subset E of X constitutes an event and

$$P_\theta(E) = \sum_{x \in E} p_\theta(x) \quad \text{or} \quad P_\theta(E) = \int_E p_\theta(x)dx.$$

The collection of all possible values of O constitute the set of all possible pure strategies of Nature and their corresponding probability distribution functions her mixed strategies.

As a result of an experiment the statistician chooses a course of action a among a collection A of all possible courses of action. The function d which associates with each outcome x of an experiment a course of action a (so that d(x) = a) is called a (statistical) decision function. By conglomerating all outcomes x associated with a fixed course of action a, the whole sample space X is then partitioned into a collection of disjoint subsets each associated with a fixed course of action. For example, in the well-known one-tailed t-test, the sample space X of all real numbers t is partitioned into two disjoint subsets, one consisting of all real numbers t less than the true value t_o and the other of all real numbers t greater than or equal to t_o. In the event that t falls within the later set, the course of action associated with it is the rejection of the null hypothesis, while when it falls on the former set, the null hypothesis is accepted. The collection D of all possible decision functions constitutes the possible strategies of the statistician. Like the set X of possible outcomes of the experiment, D is often infinite in practice.

Experiments cost time and money so that decisions based on a wrong estimate of θ may sometimes be too expensive.

Suppose that we have agreed to adopt a magnitude of this numerical loss (denoted by $\ell(a, \theta)$) which depend on the action a and parameter θ The utility payoff matrix for a statistical game is the expected value of this loss which either

$$L(d, \ ) = \sum_{x \epsilon X} \ell(d(x), \theta) p_{\theta}(x)$$

or

$$L(d, \theta) = \int_X \ell(d(x), \theta) p_{\theta}(x) dx.$$

A statistician's optimal course of action must minimize this expected loss or maximize its negative value (which is one's expected gain).

Before undertaking a more general application of the theory of statistical decision-making, let us first consider the following household example:

Juana de la Cruz is thinking of buying a new dress for the coming town fiesta. She has learned from past experience that if her husband Juan is in a good mood, then she can easily get his consent to buy a very expensive dress. On the other hand, if he is in a bad mood, he may not even agree to anything she says. Before making a final decision, Juana decided to perform an experiment: she will tell him upon coming home that the afternon paper got lost. She expects anything like one of the following statements:

x_1. "Newspapers sometimes get lost";

x_2. "I told you you should have a place to keep newspapers";

x_3. "Why did I ever get married".

Of course, the two possible states of Nature in this instance are:

s_1. Juan is in a good mood;

s_2. Juan is in a bad mood.

The three possible courses of action for Juana to take include:

s_1. Buy an expensive dress;

s_2. Buy an ordinary dress;

s_3. Do not buy any dress at all.

Considering her personal tastes and idiosyncracies, Juana thought that her utility losses may be relatively written in matrix form $\chi(a_i, s_j)$ as follows:

States of Nature

	s_1	s_2
a_1	0	-5
a_2	-1	-3
a_3	-3	-2

Based on Juana's long experience with Juan, she thought the following table of probabilities is a very good indicator of Juan's moods:

Probability of observing x_i given that $s = s_j$

$$P_{s_j}(x_i)$$

	x_1	x_2	x_3
s_1	.65	.25	.10
s_2	.20	.30	.50

How should Juana act under circumstances. Her collection of possible strategies or decision functions are tabulated as follows.

	x_1	x_2	x_3		x_1	x_2	x_3
d_1	a_1	a_1	a_1	d_{15}	a_2	a_2	a_3
d_2	a_1	a_1	a_2	d_{16}	a_2	a_3	a_1
d_3	a_1	a_1	a_3	d_{17}	a_2	a_3	a_2
d_4	a_1	a_2	a_1	d_{18}	a_2	a_3	a_3
d_5	a_1	a_2	a_2	d_{19}	a_3	a_1	a_1
d_6	a_1	a_2	a_3	d_{20}	a_3	a_1	a_2
d_7	a_1	a_3	a_1	d_{21}	a_3	a_1	a_3
d_8	a_1	a_3	a_2	d_{22}	a_3	a_2	a_1
d_9	a_1	a_3	a_3	d_{23}	a_3	a_2	a_2
d_{10}	a_2	a_1	a_1	d_{24}	a_3	a_2	a_3
d_{11}	a_2	a_1	a_2	d_{25}	a_3	a_3	a_1
d_{12}	a_2	a_1	a_3	d_{26}	a_3	a_3	a_2
d_{13}	a_2	a_2	a_1	d_{27}	a_3	a_3	a_3
d_{14}	a_2	a_2	a_2				

Many of these decision functions, it is easy to see, are quite unreasonable.

By using the relations

$$L(d_i,s_j) = \sum_{k=1}^{3} \ell(d_i(x_k),s_j)p_{s_j}(x_k)$$

and

$$R(d_i,s_j) = \max_k L(d_k,s_j) - L(d_i,s_j)$$

we obtain the following matrices of Juana's expected losses and expected regrets:

$L(d_i,s_j)$

	s_1	s_2
d_1	.00	-5.00
d_2	-.10	-4.00
d_3	-.30	-3.50
d_4	-.25	-4.40
d_5	-.35	-3.40
d_6	-.55	-2.90
d_7	-.75	-4.10
d_8	-.85	-3.10
d_9	-1.05	-2.60
d_{10}	-.65	-4.60
d_{11}	-.75	-3.60
d_{12}	-.95	-3.10
d_{13}	-.90	-4.00
d_{14}	-1.00	-3.00
d_{15}	-1.20	-2.50
d_{16}	-1.40	-3.70
d_{17}	-1.50	-2.70
*d_{18}	-1.70	-2.20
d_{19}	-1.95	-4.40
d_{20}	-2.05	-3.40
d_{21}	-2.25	-2.90
d_{22}	-2.20	-3.80
d_{23}	-2.30	-2.80
d_{24}	-2.50	-2.30
d_{25}	-2.70	-3.50
d_{26}	-2.80	-2.50
d_{27}	-3.00	-2.00

$R(d_i,s_j)$

	s_1	s_2
d_1	.00	3.00
d_2	.10	2.00
d_3	.30	1.50
d_4	.25	2.20
d_5	.35	1.40
d_6	.55	.90
d_7	.75	2.10
d_8	.85	1.10
d_9	1.05	.60
d_{10}	.65	2.60
d_{11}	.75	1.60
d_{12}	.95	1.10
d_{13}	.90	2.00
*d_{14}	1.00	1.00
d_{15}	1.20	.50
d_{16}	1.40	1.70
d_{17}	1.50	.70
d_{18}	1.70	.20
d_{19}	1.95	2.20
d_{20}	2.05	1.40
d_{21}	2.25	.90
d_{22}	2.20	1.80
d_{23}	2.30	.80
d_{24}	2.50	.30
d_{25}	2.70	1.50
d_{26}	2.80	.50
d_{27}	3.00	.00

Juana's maximin and minimax regret strategies are respectively d_{18} and d_{14}. The first strategy recommends that she buys an ordinary dress in the event that Juan answers something like "Newspapers sometimes get lost" and that she desist from buying a dress otherwise. The second minimax regret strategy advises she should buy an ordinary dress in any case. Of course, if she knows that her husband is in a good mood with a prbability p, then Juana's optimal decision would be that one which maximizes the expectation of her expected loss

$$L(d) = pL(d, s_1) + (1 - p)L(d, s_2) .$$

For instance, if the probability that Juan be in a good mood is 2/3, then d_6 would be the best decision for Juana.

A branch of statistics wherein decision theory can find a fitting expression is in quality control. Consider, for example, manufactring firm which regularly buys a certain commodity in lots of N for use as a raw material. Without any inspection procedure, the firm may accept very poor lots which consequently may lower the quality of their products to the detriment of the firm's reputation. On the other hand, consistent 100% inspection of a lot is very expensive. The problem that presents itself therefore is how to find a systematic decision procedure to use in such circumstances. Suppose c_1 is the cost of inspecting each item of a lot and c_2 is the cost of scrapping an item. Two decisions are possible: either to accept the lot (d_1) or inspect the lot 100% (d_2). Their respective costs are $N\theta c_2$ and Nc_1 where θ is the proportion of defectives. If $p(\theta)$ denotes the (a priori) probability distribution of θ and x and 1 - x are probabilities associated with decisions d_1 and d_2 respectively, then the expected value of the firm (gain) is

$$L(x,p) = -\int_0^1 (xN\theta c_2 + (1-x)Nc_1)p(\theta)d\theta$$

$$= -Nc_1 - Nx\int_0^1 (\theta c_2 - c_1)p(\theta)d\theta \, .$$

In actual practice, an approximation of $p(\theta)$ is often known from past experience, but if no such information is available, the maximin criteria recommends the mixed strategy $(x, 1-x)$ which achieves the

$$\max_x \min_p L(x,p) = \max \min \left(-Nc_1 - Nx\int_0^1 (\theta c_2 - c_1)p(\theta)d\theta\right).$$

Since the probability function $p(\theta)$ may be zero for values of such that $\theta c_2 - c_1$ is negative, then

$$\max_x \min_p L(x,p) = \max_x \min_p \left(-Nc_1 - Nx\int_{\frac{c_1}{c_2}}^1 (\theta c_2 - c_1)p(\theta)d\theta\right),$$

$$= \max (-Nc_1 - Nxc), \text{ for some positive constant } c.$$

The maximin is therefore attained when $x = 0$, that under 100% inspection of the lot. However, if $p(\theta) = k$ (a constant), then $L(x,p) = -Nc_1 - \frac{1}{2}Nkx(c_2 - 2c_1)$. Hence $\max_x \min_p L(x,p) = \max L(x,k)$, and therefore, the maximin strategy is realized by $x = 0$ when $c_2 = 2c_1$ is positive and by $x = 1$ when $c_2 = 2c_1$ is negative.

But now, suppose that a sampling inspection of n items ou tof the lot is made and to resort only to a 100% inspection when the number of defectives i is greater to or equal to some number s. The cost of of decision d_1 is then

$$nc_1 + (N - n)\theta c_2$$

and the expected value of the gain is given by

$$L(s,p) = - \int_0^1 \Big(\sum_{i=0}^{s-1} \binom{n}{i} \theta^i (1-\theta)^{n-i} (nc_1 + (N-n)\theta c_2)$$

$$(1 - \sum_{i=0}^{s-1} \binom{n}{i} \theta^i (1-\theta)^{n-i}) Nc_1 \Big) p(\theta) d\theta$$

$$= -Nc_1 - (N-n) \int_0^1 (\theta c_2 - c_1) \sum_{i=0}^{s-1} \binom{n}{i} \theta^i (1-\theta)^{n-i} p(\theta) d$$

$$= -Nc_1 - (N-n) c_2 \sum_{i=0}^{s-1} \binom{n}{i} \int_0^1 \theta^{i+1} (1-\theta)^{n-i} p(\theta) d\theta$$

$$+ (N-n) c_1 \sum_{i=0}^{s-1} \binom{n}{i} \int_0^1 \theta^i (1-\theta)^{n-i} p(\theta) d\theta .$$

If $p(\theta)$ is assumed to be some constant k and the above integration is performed, then the following expressions will be obtained:

$$L(x,p) = -Nc_1 - (N-n)kc_2 \sum_{i=0}^{s-1} \frac{i+1}{(n+1)(n+2)}$$

$$(1 - \sum_{j=0}^{i+1} \binom{n+2}{j} \theta^j (1-\theta)^{n+2-j}$$

$$+ (N-n)kc_1 \sum_{i=0}^{s-1} \frac{1}{n+1} (1 - \sum_{j=0}^{i} \binom{n+1}{j} \theta^j (1-\theta)^{n+1-j} \Big]_0^1$$

$$= -Nc_1 - \frac{(N-n)ks}{(n+1)} \left(\frac{c_2(s+1)}{2(n+1)} - c_1 \right).$$

Thus, the criterion of Laplace recommends a choice of s as close as possible to the value of

$$\frac{2c_1(n+2)}{c_2} - 1. \quad \text{For} \quad \frac{c_2}{c_1} = 25, \quad N = 1{,}000,$$

and $n = 100$, this number s is approximately 7. The advise is then to accept a lot if in a random sample of 100 from the lot there are not more than 7 defectives.

A solution which maximizes $L(x, p)$ for a given $p(\theta)$ is also called a Bayes solution. For example, if

$$p(\theta) = 25 \quad \text{for} \quad 0.01 \leq \theta \leq 0.05$$

and $p(\theta) = 0$ for all other values of θ the maximum of above expectation $L(x, p)$ or the minimum loss of the firm is achieved when $s = 7$. The problem can also be viewed in terms of minimax regret.

6. Applications to Economics.

The following is one of the many special instances of the linear programming problem applied to economics. This is often called the transportation problem.

A certain company has four warehouses and six stores. It has been so decided by the board of directors of the company to transfer 220 tons of goods from the four warehouses to the six stores as follows:

Store	Amount to be transported
1	40 tons
2	40 tons
3	60 tons
4	20 tons
5	40 tons
6	20 tons

Of the 220 tons of goods, 50 tons are coming from warehouse 1, 60 tons from warehouse 2, 20 tons from warehouse 3, and 90 tons warehouse 4. The shipping costs of one ton of the commodity from warehouse i to store j, c_{ij}, are given by the following matrix (in pesos):

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix} 90 & 120 & 90 & 60 & 90 & 100 \\ 70 & 90 & 70 & 70 & 50 & 50 \\ 60 & 50 & 90 & 110 & 30 & 110 \\ 60 & 80 & 110 & 20 & 20 & 100 \end{pmatrix} \end{array}
$$

The problem is to determine the most economical way of effecting the transfer.

Let x_{ij} denote the amount of goods from warehouse i to be sent to store j. Then, mathematically, the problem is to find the smallest value of the linear function

$$
\begin{aligned}
&90x_{11}+120x_{12}+90x_{13}+60x_{14}+90x_{15}+100x_{16}+70x_{21}+30x_{22} \\
&+70x_{23}+70x_{24}+50x_{25}+50x_{26}+60x_{31}+50x_{32}+90x_{33}+110x_{34} \\
&+30x_{35}+110x_{36}+60x_{41}+80x_{42}+110x_{43}+20x_{44}+20x_{45}+100x_{46}
\end{aligned}
$$

subject to the following conditions:

$$x_{11} + x_{21} + x_{31} + x_{41} = 40,$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 40,$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 60,$$
$$x_{14} + x_{24} + x_{34} + x_{44} = 20,$$
$$x_{15} + x_{25} + x_{35} + x_{45} = 40,$$
$$x_{16} + x_{26} + x_{36} + x_{46} = 20,$$
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} = 50,$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} = 60,$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} = 20,$$
$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} = 90.$$

There is a known iterative procedure that solves this problem by hand computation, but this is out of the question in the present discussion. The optimal solution, it can be easily verified, is given by the set of

$x_{13} = 50$, $x_{22} = 40$, $x_{26} = 20$, $x_{31} = 10$, $x_{33} = 10$, $x_{41} = 30$, $x_{44} = 20$, and $x_{45} = 40$, with all other x_{ij}'s equal to zero.

The first linear programming ever solved is called the nutrition problem. Here is a very simplified form.

The two foods rice and fish are essential components of any Filipino's diet. Each of these contains the nutrients protein measured in grams and the other measured in calories. The nutrient contents of one kilo of fish are 4,000 calories and 200 grams of protein. On the other hand, one kilo of rice contains 2,000 calories and 50 grams of protein. It is known that the daily requirements of a good diet is 3,000 calories and 100 grams of protein. It is also known that one kilo of rice costs a peso while one kilo of fish costs 2 pesos. What is the most economical combination for a good diet?

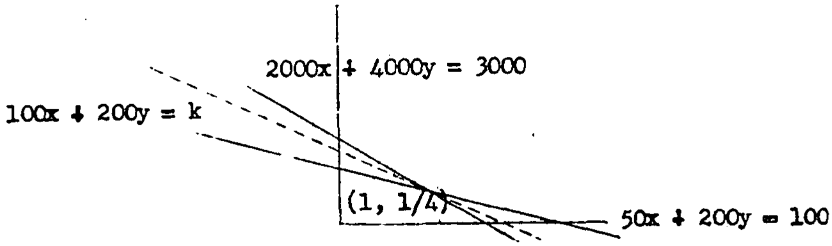
If x and y denote the amounts of rice and fish respectively needed, then the problem is to minimize the cost function 100x + 200y (in centavos) subject to the inequalities

$$x \geqq 0, \quad y \geqq 0,$$

$$2{,}000x + 4{,}000y \geqq 3{,}000$$

$$50x + 200y \geqq 100.$$

Graphically, the solution is easily read from



The solution is the point of intersection of the lines $2x + 4y = 3$ and $x + 4y = 2$, that is $x = 1$ and $y = 1/4$.

The next application is an example of what is called dynamic programming. This is a problem which involve a number (sometimes an infinite number) of decisions made in the course of time. Each decision is dependent on all past decisions and in turn affects all decisions in the future.

A man is engaged in buying and selling rice, which requires a bodega for storage. His bodega has a maximum capacity of 1,000 cavans of rice. His policy is to order at the middle of the month, for delivery in the beginning of the following month. During the month he can sell any amount of rice available in his bodega at the time. If he starts the year 1962 with 500 cavans in stock, how should he purchase and sell each month in order to maximize profits, when the cost and sales price cavan for each month are given by the following table:

Cost Prices		Sales Prices	
January 15	₱20	January	₱23
February 14	21	February	24
March 15	22	March	22
April 15	22	April	20
May 15	22	May	23
June 15	22	June	27
July 15	27	July	25
August 15	24	August	25
September 15	21	September	23
October 15	28	October	25
November 15	22	November	25
December 15	23	December	26

Let $f_n(s)$ = maximin profit that can be achieved in the remaining n months of the year when the existing stock at the time is s cavans;

x_k = the number of cavans of rice to be sold during the kth month,

y_k = the number of cavans of rice to be ordered in the 15th of the kth month;

c_k = cost of rice per cavan during the kth month;

d_k = selling price of rice per cavan during the kth month.

Then

$$f_n(s_{13-n}) = \max \{c_{13-n}x_{13-n} - d_{13-n}y_{13-n} + f_{n-1}(s_{13-n})\}.$$

$$0 \leqq x_{13-n} \leqq s_{13-n}$$

$$0 \leqq y_{13-n} \leqq 1{,}000 - (s_{13-n} - x_{13-n})$$

Based on this formula, the policies for each month are computed as follows:

December:

$$f_1(s_{12}) = \max \{ 26x_{12} - 23y_{12} \} = 26s_{12},$$

$$x_{12} = s_{12}, \quad y_{12} = 0;$$

November:

$$f_2(s_{11}) = \max \{25x_{11} - 22y_{11} + 26(s_{11}+y_{11}-x_{11})\}$$

$$= 25s_{11} + 4{,}000,$$

$$x_{11} = s_{11}, \quad y_{11} = 1{,}000;$$

October:

$$f_3(s_{10}) = \max \{ 25x_{10} - 28y_{10} + 25(s_{10}+y_{10}-x_{10}) + 4{,}000 \}$$

$$= 25s_{11} + 4{,}000,$$

$$x_{10} = a(< 1{,}000), \quad y_{10} = 0;$$

September:

$$f_4(s_9) = \max\{23x_9 - 21y_9 + 25(s_9 + y_9 - x_9) + 4{,}000\}$$
$$= 23s_9 + 8{,}000,$$
$$x_9 = s_9, \quad y_9 = 1{,}000;$$

August:

$$f_5(s_8) = \max\{25x_8 - 24y_8 + 23(s_8 + y_8 - x_8) + 8{,}000\}$$
$$= 25s_8 + 8{,}000,$$
$$x_8 = s_8, \quad y_8 = 0;$$

July:

$$f_6(s_7) = \max\{25x_7 - 27y_7 + 25(s_7 + y_7 - x_7) + 8{,}000\}$$
$$= 25s_7 + 8{,}000,$$
$$x_7 = b(< s_7), \quad y_7 = 0;$$

June:

$$f_7(s_6) = \max\{27x_6 - 25y_6 + 25(s_6 + y_6 - x_6) + 8{,}000\}$$
$$= 27s_6 + 8{,}000,$$
$$x_6 = s_6, \quad y_6 = s_7 \ (< 1{,}000);$$

May:

$$f_8(s_5) = \max\{23x_5 - 22y_5 + 27(s_5 + y_5 - x_5) + 8{,}000\}$$
$$= 23s_5 + 13{,}000,$$
$$x_5 = s_5, \quad y_5 = 1{,}000;$$

April:

$$f_9(s_4) = \max\{20x_4-22y_4+23(s_4+y_4-x_4)+13{,}000\}$$
$$= 20s_4 + 14{,}000,$$
$$x_4 = s_4, \quad y_4 = 1{,}000;$$

March:

$$f_{10}(s_3) = \max\{22x_3-22y_3+20(s_3+y_3-x_3)+14{,}000\}$$
$$= 22s_3 + 14{,}000,$$
$$x_3 = s_3, \quad y_3 = 0;$$

February:

$$f_{11}(s_2) = \max\{24x_2-21y_2+22(s_2+y_2-x_2)+14{,}000\}$$
$$= 24s \ + 15{,}000$$
$$x_2 = s_2, \quad y_2 = 1{,}000;$$

January:

$$f_{12}(s_1) = \max\{23x_1-20y_1+24(s_1+y_1-x_1)+15{,}000\}$$
$$= 23s_1 + 19{,}000,$$
$$x_1 = s_1 = 500, \quad y_1 = 1{,}000.$$

Thus the maximum profit achievable is $f_{12}(s_1)$ — ₱30,500. The profit is achieved by following program of buying and selling:

	x_k	y_k	s_k
January	500	1,000	500
February	1,000	1,000	1,000
March	1,000	0	1,000
April	0	1,000	0
May	1,000	1,000	1,000
June	1,000	c	1,000
July	b	0	c
August	c - b	0	c - b
September	0	1,000	0
October	a	0	1,000
November	1,000-a	1,000	1,000-a
December	1,000	0	1,000.

7. **Applications to Political Science.**

As a final example of decision making solved by mathematical methods, we shall consider the following problem in politics.

In a certain election year, the Liberal and Nacionalista Party conventions both agreed to let their party nominations for vice-president be decided by their respective directorates. Based on a sampling of public opinion conducted by the Bebot Statistics, Inc. the following odds were recorded for the various nominees.

Nationalista	Odds	Liberal
Sumulong	2:1	Manglapuz
Sumulong	3:2	Cases
Sumulong	3:5	Diokno
Aytona	1:2	Manglapuz
Aytona	4:1	Cases
Aytona	1:3	Diokno
Laurel	3:1	Manglapuz
Laurel	2:1	Cases
Laurel	3:7	Diokno

The political bosses of both parties have decided to select their respective candidates in accordance with the best mathematical decision procedure.

If the utility index payoff is taken as the probability of winning, then the payoff matrix for the two parties may be written as

Liberal

$$\text{Nacionalista}\quad \begin{array}{c} \\ S \\ A \\ L \end{array} \begin{array}{c} \begin{array}{ccc} \quad M \quad & \quad C \quad & \quad D \quad \end{array} \\ \begin{pmatrix} (.667,\ .333) & (.600,\ .400) & (.375,\ .625) \\ (.333,\ .667) & (.800,\ .200) & (.250,\ .750) \\ (.750,\ .250) & (.667,\ .333) & (.300,\ .700) \end{pmatrix} \end{array}$$

With this payoff matrix the game is not zero sum, but the game can be converted into a zero sum game by substracting the payoff entries of one party from the other's. For instance, the payoff of the Nacionalista party is given by

$$\begin{pmatrix} .333 & .200 & -.250 \\ -.333 & .600 & -.500 \\ .500 & .333 & .400 \end{pmatrix}$$

It is easy to see that (1,3) is a saddle point of this matrix and hence the optimal choices for the two parties are Sumulong and Diokno.

If the odds between Laurel and Diokno were interchanged so that the payoff matrix becomes

$$\begin{pmatrix} .333 & .200 & -.250 \\ -.333 & .600 & -.500 \\ .500 & .333 & -.400 \end{pmatrix}$$

then there is no longer a saddle point strategy and the game has to be resolved by a randomization of strategies. It is clear that the probabilities of Laurel respectively dominate the probabilities of Sumulong and hence without loss of generality Sumulong may be dropped off from the nomination race. The resulting new matrix is then

$$\begin{pmatrix} -.333 & .600 & -.500 \\ .500 & .333 & .400 \end{pmatrix}$$

Again, the probabilities of Diokno which are the negatives of the third column respectively dominate the probabilities of Manglapuz which are the negatives of the first column and hence Manglapuz may be dropped also in the race. The remaining payoff matrix is

$$\begin{pmatrix} .600 & -.500 \\ .333 & .400 \end{pmatrix}.$$

Now, if (x, 1-x) and (y, 1-y) respectively denote the mixed strategies of he Nacionalista and Liberal partıes, then the optimal solutions are found by solving the following inequalities:

$$.600x + .333(1 - x) \geqq v, \quad .600y - .500(1 - y) \leqq v,$$
$$-.500x + .400(1 - x) \geqq v, \quad .333y + .400(1 - y) \leqq v.$$

The actual solutions are obtained by simply considering the case when all inequalities are equalities: they are $(\frac{67}{1167}, \frac{1100}{1167})$ and $(\frac{900}{1167}, \frac{267}{1167})$ The value of the game is $\frac{4065}{11670}$ in favor of the Nationalista Party. The mixed strategies weigh heavier for Aytona and Cases.